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# A branch-and-cut algorithm for the partitioning-hub location-routing problem

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ABSTRACT

We introduce the Partitioning-Hub-Location-Routing Problem (PHLRP), a hub location problem involving graph partitioning and routing features. The PHLRP consists of partitioning a given network into sub-networks, locating at least one hub in each sub-network and routing the traffic within the network at minimum cost. This problem finds applications in deployment of an Internet Routing Protocol called Intermediate System–Intermediate System (ISIS), and strategic planning of LTL ground freight distribution systems. We present an Integer Programming (IP) model for solving exactly the PHLRP and explore possible valid inequalities to strengthen it. Computational experiments prove the effectiveness of our model which is able to tackle instances of PHLRP containing up to 20 vertices.



## 1. Introduction

A common problem in many real life applications consists of transferring, as efficiently as possible, a set of commodities (e.g., goods, people, data packages) between source–destination pairs in a network (e.g., a rail network or an Internet Protocol network). Usually, specific vertices in the network are selected to aggregate the traffic flow corresponding to several source–destination pairs. These vertices are called *hubs* and induce a *backbone* of transfers throughout the whole network. Hubs facilitate the transfers and optimize the costs, for this reason their localization in the network plays a central role.

The literature on hub location describes several versions of the hub location problem, mainly differing from one another by the type of the objective function considered, the number and nature of the required hubs, and the presence (absence) of hub capacities (see [1–3], for an introduction and recent surveys). In this paper we investigate a peculiar version of the hub location problem, called *the Partitioning-Hub-Location-Routing Problem* (PHLRP). Specifically, the PHLRP consists of partitioning a given network into sub-networks, locating at least one hub in each sub-network, and routing the traffic within the network at minimum cost. The PHLRP involves both graph partitioning and routing features, and mainly arises from the deployment of an Internet routing protocol called Intermediate System–Intermediate System (ISIS), see Retana and White [4] and Özsoy et al. [5].

The PHLRP can be described as follows. Assume that a network (a digraph) $D=(V,A)$ and a companion graph $G=(V,E)$ are given, and that the arc set $A$ is symmetric, i.e., for all $i,j \in V$ both arcs $(i,j)$ and $(j,i)$ belong to $A$. Assume also that for each pair of arcs $(i,j),(j,i) \in A$, $i<j$, there exists an edge $\{i,j\} \in E$ with $i<j$. Let $V_1,V_2,\dots,V_k$ be disjoint subsets of $V$ such that $V=\bigcup_l V_l$, and denote $(V_1,V_2,\dots,V_k)$ as the corresponding partitioning of $V$ into subsets. Assume that $F_L \le |V_l| \le F_U$, $l=1,2,\dots,k$, where $F_L$ and $F_U$ are some lower and upper bound values on the sizes of subsets. In accordance with the ISIS terminology we define an *area* $D_l=(V_l, A_l)$ as the sub-network of $D$ induced by $V_l$, $l=1,2,\dots,k$ (see Fig. 1(b)). The use of the parameters $F_L$ and $F_U$ provides a control over the desired sizes and the number of areas in the partition and prevents the presence of either very large areas or very large number of small areas which could give rise to stability and convergency problems in the ISIS protocol (see [4, Chapter 3]). We denote $H$ as a subset of hub vertices in $V$ and we assume that $|H| \le Y$, where $Y$ is an upper bound on the number of hubs we want to locate in the network. We call the vertices in $V-H$ the *local vertices* (see Fig. 1(c)) and the sub-network of $D$ induced by $H$ *the backbone*, denoted by $D_H=(H,A_H)$.

Let $T$ be a set containing all source–sink pairs which have traffic in-between. For each pair $(u,v) \in T$, we call vertex $u$ the *source* and vertex $v$ the *sink*. Let $d_{u,v}$ be the amount of required traffic flow between the source–sink pair $(u,v) \in T$. A non-negative cost $c_{i,j}^{u,v}$ is incurred when unit flow associated with the pair $(u,v) \in T$ passes through the arc $(i,j) \in A$. In accordance with the ISIS protocol (see [5]), we assume that routing of traffic between any

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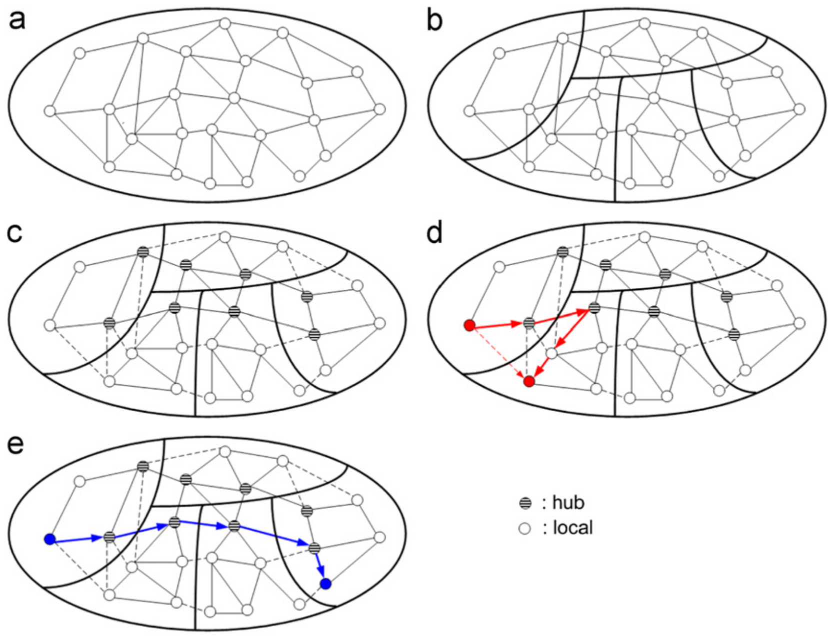


**Fig. 1.** An example of partitioning, hub location and routing in a network. (a) A network. (b) Partitioning of the network into areas. (c) Hub location on the network. (d) An example of inter-area routing. (e) An example of an alternative inter-area routing.

source–destination pair in $T$ complies with the following assumptions:

(A1) Inter-area traffic flows can be sent through an arc $(i,j)$ only if $(i,j)$ is on the backbone (i.e., $(i,j) \in A_H$).
(A2) Intra-area traffic is realized over paths which entirely lie within the areas.

The first assumption implies that two local vertices in different areas cannot exchange traffic flows over an arc that lies in-between. Local vertices in different areas can send flows to each other only through the hubs located within their areas. We illustrate this fact in Fig. 1(c) by displaying in dashed lines the inter-area arcs which are adjacent to local vertices. It is worth noting that assumptions (A1) and (A2) imply that:

1. an area can exchange flows with the other areas only if it possesses a hub vertex;
2. hubs of areas with traffic demand in-between have to be connected to each other over the backbone;
3. if the source and the sink of a traffic flow are in different areas, then the flow is first directed towards the hub within the source area; subsequently, it is sent over the backbone towards the hub of the sink area; finally, it is directed from this hub towards the sink;
4. if the source and the sink are in the same area, then the flow is simply sent over a path that is entirely contained within the area.

Fig. 1(d) and (e) show two examples of inter-area flows. Note that in Fig. 1(d), the source and the destination are in different areas and although there exists an arc in-between (shown dashed), the flow is sent through the hubs. Then, given the network $D=(V,A)$, the Partitioning-Hub-Location-Routing Problem consists of (i) partitioning $D$ into areas $D_1,D_2,\dots,D_k$ of size at least $F_L$ and at most $F_U$, (ii) determining the hubset $H$, (iii) routing the traffic for each source–sink pair $(u,v) \in T$ in accordance with assumptions (A1) and (A2), (iv) minimizing the overall routing cost between source–sink pairs in $T$.

Interestingly, the nature of the PHLRP consists of multiple aspects, namely: a graph (or network) partitioning aspect, a routing aspect, and a hub location aspect. The graph partitioning aspect can be treated as a *size constrained graph partitioning problem*, already investigated in the literature by Özsoy and Labbé [6,7]. The routing aspect can be modeled by using multi-commodity flow variables and constraints (i.e., flow balance constraints). On the contrary, the hub location aspect cannot be treated as a classic hub location problem described in the literature. In fact, according to Alumur and Kara [3], hub location problems are generally based on three prevalent assumptions:

(B1) the hub network is complete with a link between every hub pair;
(B2) cost of transfer between the hubs is less than the cost of transfer between non-hub vertices and hub vertices (due to economies of scale);
(B3) no direct transfer between two non-hub vertices is allowed.

In the PHLRP only assumption (B2) is kept, while assumption (B1) does not hold, and assumption (B3) is imposed in a weaker way. In fact, ISIS protocol allows transfers between non-hub vertices in order to decrease significantly transfer costs. In order to provide exact solutions to instances of the PHLRP in the next section we shall present a possible mixed integer programming formulation for the problem.

## 2. An IP formulation for the PHLRP

A possible approach for modeling the PHLRP consists of considering two replicates of the input digraph $D=(V,A)$ (see Fig. 2(a)). We refer to these replicates as *layers*. The sets of vertices and the arcs in the two layers are identical, and between corresponding vertices of the layers we introduce cost-free and uncapacitated inter-layer arcs. Namely, there is an inter-layer arc from vertex $i$ of Layer 1 to vertex $i$ of Layer 2 and vice versa for each $i$ in $V$.

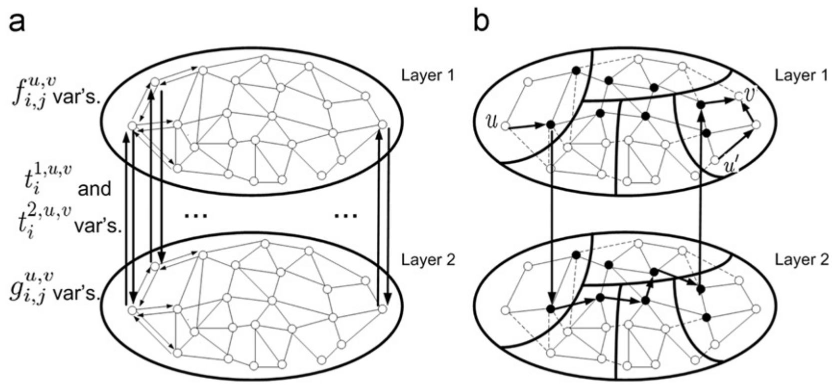


**Fig. 2.** Illustrating Layers 1 and 2 in Formulation (1). (a) The flow variables in the formulation. For the sake of simplicity, only some of the intra-layer and inter-layer arcs are displayed. Each of the undirected links within the layers stands for two arcs in opposite directions. (b) Examples of an intra-area flow (from $u'$ to $v$) and an inter-area flow (from $u$ to $v$).

In this approach, Layer 1 accommodates the flows within the areas (i.e., all the intra-area flows, and the portions of the inter-area flows that lie within the areas) and Layer 2 accommodates the flows on the backbone. Specifically, we impose that if the source and the sink vertices, say $(u,v)$, lie in the same area then the flow stays within Layer 1 all the way from the source to the sink. Otherwise, if $u$ and $v$ lie in different areas, then the flow starts at the source vertex $u$ in Layer 1, and continues within this layer until it arrives at a hub. There, it jumps onto Layer 2 through the inter-layer arc associated with the hub node. It continues in Layer 2 until it arrives at a hub of the sink area. Then, it jumps back onto Layer 1 and arrives at the destination $v$ in Layer 1 (see Fig. 2(b)). We impose that in Layer 1 no flow can pass through arcs that lie between vertices belonging to different areas. Moreover, we also impose that an inter-area flow can pass from one layer to another only through an inter-layer arc corresponding to a hub.

Define the partitioning variables $w_{u,v} \in \{0,1\}$, for all $u,v \in V$ such that $u < v$, as

$$w_{u,v} = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are in the same area} \\ 0 & \text{otherwise} \end{cases}$$

Define the hub-location variables $x_u \in \{0,1\}$, for all $u \in V$, as

$$x_u = \begin{cases} 1 & \text{if vertex } u \text{ is designated as a hub} \\ 0 & \text{otherwise} \end{cases}$$

Let $f_{i,j}^{u,v}$ and $g_{i,j}^{u,v}$, for all $(u,v) \in T$ and $(i,j) \in A$, be the flows in Layers 1 and 2, respectively, associated with the source–sink pair $(u,v) \in T$ on the arc $(i,j) \in A$. Moreover, let $t_i^{1,u,v}$ and $t_i^{2,u,v}$, for all $(u,v) \in T$ and $i \in V$, be the inter-layer flows from Layer 1 to Layer 2 and vice versa, respectively. Specifically, $t_i^{1,u,v}$ denotes the flow associated with the source–sink pair $(u,v) \in T$ on the inter-layer arc $(i,i)$ from Layer 1 to Layer 2 (i.e., the arc from vertex $i$ in Layer 1 to its counterpart in Layer 2). Similarly, $t_i^{2,u,v}$ denotes the flow on the arc in the reverse direction (i.e., the arc from vertex $i$ in Layer 2 to the vertex $i$ in Layer 1). We denote $\delta^+(i)$ and $\delta^-(i)$ as the outgoing and ingoing arcs of a vertex $i$ in $D$, respectively, i.e., $\delta^+(i) = \{(l_1,l_2) \in A : l_1 = i\}$ and $\delta^-(i) = \{(l_1,l_2) \in A : l_2 = i\}$. Then, a possible IP formulation for the PHLRP can be stated as follows:

$$\min \sum_{(u,v) \in T} \sum_{(i,j) \in A} c_{i,j}^{u,v} d_{u,v} (f_{i,j}^{u,v} + g_{i,j}^{u,v}) \tag{1a}$$

$$w_{u,v} + w_{u,t} - w_{v,t} \le 1 \quad \forall u,v,t \in V : u \ne v, u \ne t, v \ne t \tag{1b}$$

$$\sum_{v \in V : u < v} w_{u,v} + \sum_{v \in V : u > v} w_{v,u} \le F_U - 1 \quad \forall u \in V \tag{1c}$$

$$\sum_{v \in V : u < v} w_{u,v} + \sum_{v \in V : u > v} w_{v,u} \ge F_L - 1 \quad \forall u \in V \tag{1d}$$

$$\sum_{u \in V} x_u \le Y \tag{1e}$$

$$\sum_{(k,j) \in \delta^+(i)} f_{k,j}^{u,v} - \sum_{(k,j) \in \delta^-(i)} f_{k,j}^{u,v} + t_i^{1,u,v} - t_i^{2,u,v} = \begin{cases} 1 & \text{if } i = u \\ 0 & \text{if } i \in V - \{u,v\} \\ -1 & \text{if } i = v \end{cases} \quad \forall (u,v) \in T \tag{1f}$$

$$\sum_{(k,j) \in \delta^+(i)} g_{k,j}^{u,v} - \sum_{(k,j) \in \delta^-(i)} g_{k,j}^{u,v} - t_i^{1,u,v} + t_i^{2,u,v} = 0, \quad \forall i \in V, \ \forall (u,v) \in T \tag{1g}$$

$$t_i^{1,u,v} \le x_i \quad \forall (u,v) \in T, \ \forall i \in V \tag{1h}$$

$$t_i^{2,u,v} \le x_i \quad \forall (u,v) \in T, \ \forall i \in V \tag{1i}$$

$$\sum_{(k,j) \in \delta^+(i)} g_{k,j}^{u,v} \le x_i \quad \forall (u,v) \in T, \ \forall i \in V \tag{1j}$$

$$f_{i,j}^{u,v} + f_{j,i}^{u,v} \le w_{i,j} \quad \forall (u,v) \in T, \ \forall (i,j) \in E \tag{1k}$$

The objective function (1a) stands for the total cost of routing the traffic within the network. Constraints (1b), also called triangle inequalities, ensure the valid partitioning of the network into areas. These constraints, first introduced by Grötschel and Wakabayashi [8] for the clique partitioning problem, guarantee that if two edges of a triangle lie *within* an area, the third edge lies within that same area too. Note that, for sake of simplicity, in the model we just provide only one of the corresponding three triangular inequalities for a fixed distinct triplet $u$, $v$, and $t$. Constraints (1c) and (1d) impose upper and lower bounds, $F_U$ and $F_L$, respectively, on the sizes of the areas. Constraint (1e) imposes an upper bound $Y$ on the number of hubs that can be located in the network. Constraints (1f) and (1g) are the flow balance equations of Layers 1 and 2, respectively. Constraints (1h) and (1i) ensure that an inter-area flow associated with a source–sink pair $(u,v)$ gets onto the backbone at a hub vertex which lies within the same area as $u$ and gets off the backbone at a hub vertex which lies within the same area as $v$. The inter-area flow from $u$ to $v$ is allowed to pass from one layer to the other over a hub vertex only. Note that, constraints (1h) and (1i) could be replaced by a stronger constraint $t_i^{1,u,v} + t_i^{2,u,v} \le x_i$, for all $(u,v) \in T$, $i \in V$. However, in preliminary experiments we have observed that the weaker constraints (1h) and (1i) provide better runtime performances. This is possibly due to the fact that mixed integer programming solvers can take advantage of the peculiar network structure provided by (1h) and (1i) which are in the form of simple arc-upper bound constraints. Constraints (1j) guarantee that the backbone is composed only of hubs. Specifically, in (1j), if a vertex is not a hub then it does not receive any flow from the other vertices in Layer 2. Constraints (1k) ensure that the flows in Layer 1 do not traverse area borders. Note that constraints (1f), (1j) and (1k) ensure that assumptions (A1) and (A2) hold.

It is easily seen that the flow variables $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$, $t_i^{1,u,v}$ and $t_i^{2,u,v}$ would always take on integral values (i.e., 0 or 1) in an optimal solution to the PHLRP. Moreover, note that assumption (B2) of general hub location problems (i.e., transfers on the backbone costs less per unit flow), can also be accommodated by replacing the cost coefficient of $g_{i,j}^{u,v}$ variables in the objective function. Finally, note that the set of constraints used in the formulation allow $t^{1,u,v}$ and $t^{2,u,v}$ variables to be positive when the source–sink pair $(u,v)$ lies in the same area. In turn, this implies multiple cost-equivalent routes for $(u,v)$: one route within Layer 1 and other routes starting at $u$ in Layer 1, passing through Layer 2, and ending at $v$ in Layer 1. This multiplicity of routes with the same cost could be avoided by adding constraints $\sum_{i \in V} t_i^{1,u,v} \le 1 - w_{u,v}$, for all $(u,v) \in T$. However, we observed a deterioration of the runtime performances when adding this constraint in preliminary experiments. The reason for this behavior is possibly due to the fact that, in this case, mixed integer programming solvers may fail in exploiting the peculiar network flow structure of the formulation.

In the next section we shall present a number of possible valid inequalities to strengthen the formulation.

**Table 1**
List of the seven implementations of formulation (1) used in the experiments.

| Implementation ID | Description |
|---|---|
| 1 | Formulation (1) with Xpress proprietary cuts |
| 2 | Formulation (1) with valid inequalities (2) |
| 3 | Formulation (1) with valid inequalities (3) |
| 4 | Formulation (1) with valid inequalities (4) |
| 5 | Formulation (1) with valid inequalities (5) |
| 6 | Formulation (1) with valid inequalities (2)–(5) |
| 7 | Formulation (1) with Xpress proprietary cuts and valid inequalities (2)–(5) |

## 3. Valid inequalities

Formulation (1) can be strengthened by means of valid inequalities. Before describing them, we introduce some preliminary notation that we will prove useful throughout the section. We denote by $K_V=(V,E_V)$ the complete graph defined over the vertex set $V$. Let $Q \subseteq V$ and $S \subseteq V$. We denote by $E(Q)$ the set of all edges in $K_V$ whose both end-vertices are in $Q$, i.e., $E(Q)=\{\{i,j\} \in E_V : i,j \in Q\}$. We denote by $E(Q,S)$ the set of edges with one end-vertex in $Q$ and the other end-vertex in $S$, i.e., $E(Q,S)=\{\{i,j\} \in E_V : i \in Q, j \in S\}$. For ease of notation, given an edge $e=\{i,j\} \in E_V$, we use $w_e$ in place of $w_{i,j}$ and, for any subset $F \subseteq E_V$, we define $w(F)=\sum_{e \in F} w_e$. We refer to the polytope defined by (1b)–(1d) as *the Size-Constrained Graph Partitioning* (SCGP) *polytope* and we observe that, since it is defined on the subspace of binary $w_{u,v}$ variables, strong valid inequalities for the SCGP would constitute strong valid inequalities for formulation (1) as well (see [6,7]).

*The* 2-*partition inequalities*: Let $S,T$ be two disjoint subsets of $V$. Then, the following inequality, called *the* 2-*partition inequality*, is valid for the PHLRP:

$$w(E(S,T)) - w(E(S)) - w(E(T)) \le \min\{|S|,|T|\} \quad \text{for } S,T \subset V : S \cap T = \emptyset \tag{2}$$

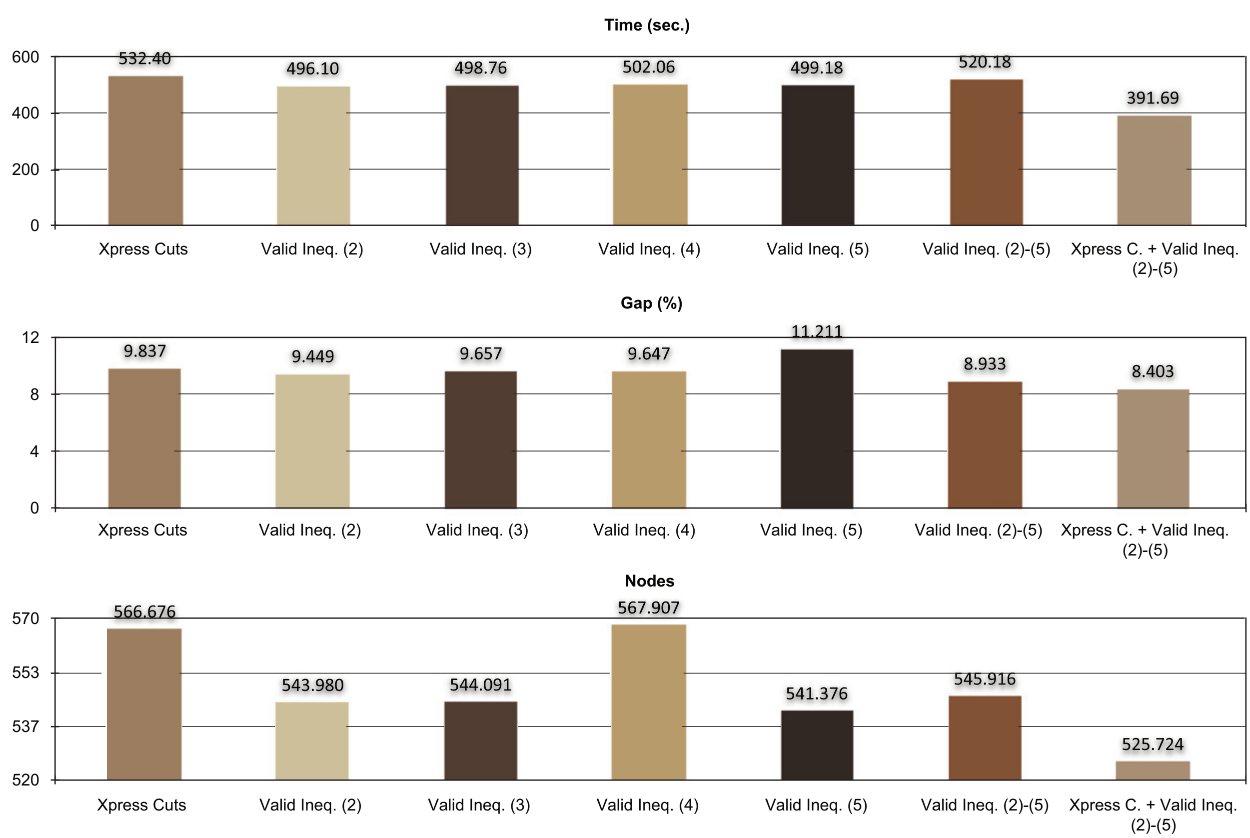


**Fig. 3.** Average performances of the seven implementations of formulation (1).

The inequality was first introduced by Grötschel and Wakabayashi [8] for the Clique Partitioning Polytope (CPP), i.e., the polytope defined by (1b). Özsoy and Labbé [7] further investigated the 2-partition inequalities and stated the conditions for which they are facet defining for SCGP polytope, i.e., the polytope defined by (1b).

*The lower general clique inequalities*: Chopra and Rao [9] introduced this family of valid inequalities for the graph partitioning problem. Özsoy and Labbé [7] adapted them to the SCGP polytope as follows. Let $Q$ be a subset of $V$ such that $|Q| > \lfloor |V|/F_L \rfloor$ and $(|Q| \bmod \lfloor |V|/F_L \rfloor) \neq 0$. Moreover, let $k$, $r$, $p$ and $q$ be defined as follows: $k = \lfloor |V|/F_L \rfloor$; $r = |V| \bmod F_L$, i.e., $|V| = kF_L + r$; $p = \lfloor |Q|/k \rfloor$ $(p \geq 1)$; and $q = (|Q| \bmod k)$ $(q > 0)$. Then, the following inequality is valid and facet defining (see Özsoy and Labbé [7]) for the SCGP polytope:

$$w(E(Q)) \geq (k-q)\binom{p}{2} + q\binom{p+1}{2} \qquad (3)$$

*The upper general clique inequalities*: Let $Q \subseteq V$ and consider the values $\phi_i^U$ computed as follows:

$$\phi_i^U = \max\left\{ \phi \in \mathbb{Z}_+ \,|\, F_L \leq \phi \leq F_U, \left\lfloor \frac{R_{i-1}-\phi}{F_L} \right\rfloor \geq \left\lceil \frac{R_{i-1}-\phi}{\phi} \right\rceil \right\}, \quad i = 1, \ldots, \left\lceil \frac{|V|}{F_U} \right\rceil$$

where $R_0 = |V|$ and $R_i = R_{i-1} - \phi_i^U$. Moreover, let $k_Q = \max\{i \,|\, \sum_{l=1}^{i} \phi_l^U \leq |Q|\}$ and $n_Q = |Q| - \sum_{l=1}^{k_Q} \phi_l^U$. Then, by adopting the convention that $\binom{0}{2} = \binom{1}{2} = 0$, the following inequality is valid and facet defining for the SCGP polytope (see Özsoy and Labbé [7]):

$$w(E(Q)) \leq \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2} \quad \forall Q \subseteq V \qquad (4)$$

**Table 2**
Performances of the seven implementations of formulation (1) with respect to the solution time (part 1).

| $\|V\|$ | Ed | Dd | Time (s) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st cut set: Xpress cuts | | | 2nd cut set: valid ineq. (2) | | | 3rd cut set: valid ineq. (3) | | | 4th cut set: valid ineq. (4) | | |
| | | | Average | Max | Min | Average | Max | Min | Average | Max | Min | Average | Max | Min |
| 9 | 0.3 | 0.3 | 0.32 | 0.92 | 0.03 | 0.46 | 0.77 | 0.02 | 0.70 | 2.09 | 0.25 | 0.34 | 0.75 | 0.14 |
| 9 | 0.3 | 0.6 | 0.77 | 2.02 | 0.02 | 0.47 | 1.45 | 0.02 | 0.66 | 1.50 | 0.02 | 0.59 | 1.66 | 0.02 |
| 9 | 0.3 | 1.0 | 1.10 | 2.23 | 0.00 | 1.24 | 2.78 | 0.00 | 1.21 | 2.97 | 0.00 | 0.91 | 2.13 | 0.00 |
| 9 | 0.6 | 0.3 | 1.01 | 2.47 | 0.02 | 1.08 | 3.19 | 0.14 | 0.98 | 2.16 | 0.02 | 0.87 | 1.75 | 0.02 |
| 9 | 0.6 | 0.6 | 2.82 | 5.06 | 1.28 | 1.59 | 3.80 | 0.58 | 2.24 | 3.78 | 0.84 | 1.51 | 2.67 | 0.59 |
| 9 | 0.6 | 1.0 | 5.58 | 9.92 | 1.22 | 2.20 | 5.84 | 0.47 | 3.47 | 6.89 | 0.91 | 2.23 | 8.02 | 0.27 |
| 9 | 0.9 | 0.3 | 3.12 | 7.42 | 0.73 | 0.88 | 3.58 | 0.20 | 2.37 | 7.99 | 0.75 | 1.48 | 4.53 | 0.49 |
| 9 | 0.9 | 0.6 | 5.75 | 9.36 | 0.03 | 1.19 | 2.58 | 0.02 | 2.55 | 4.09 | 0.17 | 1.07 | 2.89 | 0.11 |
| 9 | 0.9 | 1.0 | 10.42 | 21.22 | 3.77 | 2.41 | 4.39 | 0.67 | 4.69 | 7.74 | 2.25 | 2.53 | 5.80 | 0.72 |
| 10 | 0.3 | 0.3 | 2.07 | 4.80 | 0.03 | 1.10 | 2.69 | 0.02 | 1.48 | 2.99 | 0.14 | 1.09 | 2.34 | 0.02 |
| 10 | 0.3 | 0.6 | 3.81 | 6.66 | 0.48 | 2.04 | 4.67 | 0.94 | 3.31 | 11.47 | 0.49 | 1.61 | 3.39 | 0.41 |
| 10 | 0.3 | 1.0 | 7.37 | 17.00 | 1.97 | 3.91 | 13.17 | 1.13 | 6.98 | 32.72 | 0.56 | 4.43 | 22.70 | 0.58 |
| 10 | 0.6 | 0.3 | 5.85 | 9.86 | 1.63 | 2.22 | 3.66 | 0.88 | 3.06 | 6.14 | 0.78 | 1.34 | 2.33 | 0.50 |
| 10 | 0.6 | 0.6 | 7.94 | 12.78 | 4.14 | 4.58 | 13.94 | 1.73 | 3.80 | 7.75 | 0.89 | 3.62 | 15.06 | 0.77 |
| 10 | 0.6 | 1.0 | 17.87 | 38.83 | 5.97 | 11.13 | 29.83 | 2.84 | 9.00 | 22.56 | 2.19 | 9.81 | 45.44 | 1.17 |
| 10 | 0.9 | 0.3 | 7.30 | 16.50 | 1.84 | 3.14 | 6.77 | 1.17 | 2.66 | 7.05 | 1.31 | 3.24 | 12.41 | 0.42 |
| 10 | 0.9 | 0.6 | 14.90 | 37.22 | 5.95 | 7.24 | 21.55 | 1.00 | 4.36 | 9.86 | 1.08 | 5.68 | 15.70 | 1.44 |
| 10 | 0.9 | 1.0 | 28.84 | 63.47 | 7.17 | 12.05 | 58.64 | 1.11 | 18.18 | 96.09 | 1.19 | 15.89 | 90.03 | 0.97 |
| 12 | 0.3 | 0.3 | 8.98 | 31.31 | 1.30 | 5.82 | 13.77 | 1.08 | 8.64 | 18.94 | 1.02 | 4.65 | 13.67 | 0.30 |
| 12 | 0.3 | 0.6 | 12.05 | 32.17 | 3.78 | 2.89 | 5.97 | 1.39 | 8.57 | 24.41 | 2.53 | 4.66 | 9.85 | 0.91 |
| 12 | 0.3 | 1.0 | 14.16 | 28.72 | 0.27 | 18.66 | 69.39 | 0.66 | 15.29 | 72.80 | 1.95 | 9.08 | 29.97 | 1.28 |
| 12 | 0.6 | 0.3 | 11.11 | 58.92 | 0.31 | 6.80 | 38.80 | 0.41 | 12.81 | 83.28 | 0.25 | 4.90 | 23.59 | 0.11 |
| 12 | 0.6 | 0.6 | 38.03 | 127.35 | 13.36 | 17.54 | 44.02 | 1.69 | 24.78 | 77.27 | 2.95 | 28.18 | 90.02 | 2.30 |
| 12 | 0.6 | 1.0 | 66.09 | 150.69 | 8.58 | 38.73 | 135.08 | 2.52 | 32.33 | 109.78 | 4.56 | 35.75 | 153.24 | 2.39 |
| 12 | 0.9 | 0.3 | 12.86 | 46.69 | 1.89 | 6.82 | 18.92 | 1.25 | 7.65 | 18.89 | 2.08 | 11.14 | 32.63 | 3.47 |
| 12 | 0.9 | 0.6 | 30.40 | 53.22 | 10.00 | 15.02 | 47.14 | 2.56 | 19.12 | 45.31 | 5.20 | 19.64 | 38.38 | 5.61 |
| 12 | 0.9 | 1.0 | 59.98 | 141.27 | 2.36 | 20.56 | 76.22 | 2.19 | 32.72 | 240.64 | 0.61 | 34.80 | 201.14 | 1.77 |
| 15 | 0.3 | 0.3 | 33.73 | 71.69 | 14.03 | 11.71 | 42.44 | 2.66 | 13.71 | 29.58 | 3.20 | 17.97 | 33.57 | 6.78 |
| 15 | 0.3 | 0.6 | 54.08 | 84.46 | 12.25 | 41.94 | 142.66 | 8.08 | 55.08 | 153.33 | 10.23 | 55.53 | 120.45 | 11.84 |
| 15 | 0.3 | 1.0 | 144.45 | 388.97 | 9.72 | 107.36 | 341.47 | 14.39 | 132.93 | 358.39 | 11.31 | 92.74 | 302.74 | 12.95 |
| 15 | 0.6 | 0.3 | 48.23 | 146.92 | 10.84 | 30.81 | 87.53 | 12.42 | 62.95 | 169.94 | 10.84 | 41.59 | 184.21 | 8.88 |
| 15 | 0.6 | 0.6 | 73.02 | 167.67 | 11.06 | 46.13 | 112.72 | 14.55 | 56.00 | 163.56 | 14.33 | 36.85 | 147.25 | 6.99 |
| 15 | 0.6 | 1.0 | 234.64 | 929.85 | 41.06 | 222.61 | 1019.03 | 27.66 | 187.32 | 917.88 | 27.06 | 136.60 | 784.69 | 11.13 |
| 15 | 0.9 | 0.3 | 122.53 | 373.75 | 21.38 | 87.82 | 287.16 | 20.73 | 127.23 | 304.63 | 13.25 | 60.51 | 212.79 | 8.22 |
| 15 | 0.9 | 0.6 | 508.47 | 3599.76 | 41.36 | 473.41 | 3599.93 | 17.33 | 313.59 | 2018.29 | 14.88 | 340.83 | 2386.35 | 11.33 |
| 15 | 0.9 | 1.0 | 204.51 | 722.67 | 34.22 | 284.03 | 798.24 | 55.97 | 286.92 | 812.31 | 27.88 | 268.03 | 828.57 | 33.42 |
| 20 | 0.3 | 0.3 | 777.55 | 3599.83 | 21.42 | 812.37 | 2444.62 | 17.63 | 933.47 | 3599.06 | 10.95 | 769.55 | 3599.64 | 17.00 |
| 20 | 0.3 | 0.6 | 2698.56 | 3599.86 | 209.22 | 2732.19 | 3599.97 | 238.17 | 2501.49 | 3600.11 | 260.89 | 2672.87 | 3599.94 | 240.11 |
| 20 | 0.3 | 1.0 | 2325.85 | 3599.68 | 165.32 | 2082.02 | 3599.94 | 56.75 | 1727.37 | 3599.91 | 42.75 | 2274.20 | 3599.93 | 112.99 |
| 20 | 0.6 | 0.3 | 1329.04 | 3599.53 | 54.56 | 1159.43 | 3599.58 | 96.24 | 832.77 | 3599.03 | 40.52 | 1204.35 | 3599.64 | 38.22 |
| 20 | 0.6 | 0.6 | 2664.42 | 3600.04 | 102.67 | 2502.63 | 3600.07 | 93.17 | 2394.14 | 3600.11 | 75.17 | 2499.41 | 3599.83 | 74.84 |
| 20 | 0.6 | 1.0 | 3371.46 | 3599.83 | 1320.56 | 3136.07 | 3600.05 | 860.57 | 3362.26 | 3600.54 | 1226.28 | 3364.53 | 3600.03 | 1248.83 |
| 20 | 0.9 | 0.3 | 2583.07 | 3600.05 | 172.50 | 2214.88 | 3600.08 | 166.67 | 2307.30 | 3599.85 | 169.67 | 2085.34 | 3599.74 | 49.17 |
| 20 | 0.9 | 0.6 | 2937.88 | 3599.93 | 715.91 | 2717.52 | 3600.10 | 740.53 | 3316.32 | 3599.99 | 766.41 | 2972.07 | 3599.99 | 407.29 |
| 20 | 0.9 | 1.0 | 3466.21 | 3600.05 | 2265.20 | 3469.87 | 3600.13 | 2301.82 | 3599.76 | 3600.08 | 3599.11 | 3488.88 | 3599.98 | 2501.92 |

*The cycle inequalities*: Conforti et al. [10] introduced the cycle inequalities for the equipartition polytope. Such inequalities can be stated as follows. Let $G_C=(V_C, E_C)$ be a cycle in $G$, then

$$w(E_C) \le |V_C|-2 \quad (5)$$

is valid for the equipartition polytope and facet defining when $|V|$ is odd and $|V_C| = \lceil |V|/2 \rceil + 1$. Özsoy and Labbé [7] further investigated the cycle inequalities and stated the conditions for which they are facet defining for the SCGP polytope.

## 4. Experiments

In this section we show the results of our computational experiments carried out on a set of randomly generated instances of the PHLRP. Specifically, we considered instances having 9, 10, 12, 15 and 20 vertices, respectively. For each value of network size $|V|$ we considered different values for the edge density $\text{Ed} = |E|/\binom{|V|}{2}$, namely 0.3, 0.6, and 0.9, respectively. Similarly, we considered three possible values for the demand density $\text{Dd} = |T|/\binom{|V|}{2}$ equal to 0.3, 0.6, and 1.0, respectively. For each possible assignment of network size $|V|$, Ed and Dd, we generated 10 random instances of PHLRP, leading to an overall number of 450 instances of the problem, downloadable at http://homepages.ulb.ac.be/~dacatanz/instancesPHLRP.zip.

We implemented formulation (1) by means of Mosel 2.0 of Xpress-MP, Optimizer version 18, running on a 2.8 GHz Intel Core i7 CPU, equipped with 3 GByte RAM and Windows Vista operating system. The Xpress-MP primal heuristic was turned on and the maximum running time was limited to 1 h per instance.

We implemented seven versions of formulation (1), each one employing a different cut set (see Table 1). In the 1st version we relied only on Xpress proprietary cuts. In the 2nd, 3rd, 4th and 5th versions we deactivated Xpress proprietary cuts and used the valid inequalities (2), (3), (4) and (5), respectively. In the 6th version we used valid inequalities (2)–(5) jointly while keeping

**Table 3**
Performances of the seven implementations of formulation (1) with respect to the solution time (part 2).

| $\|V\|$ | Ed | Dd | Time (s) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 5th cut set: valid ineq. (5) | | | 6th cut set: valid ineq. (2)–(5) | | | 7th cut set: Xpress Cuts+V.I. (2)–(5) | | | Among 1st–7th | | Among 2nd–6th | |
| | | | Average | Max | Min | Average | Max | Min | Average | Max | Min | Best | Worst | Best | Worst |
| 9 | 0.3 | 0.3 | 0.16 | 0.30 | 0.02 | 0.42 | 0.83 | 0.22 | 0.29 | 0.80 | 0.03 | 5 | 3 | 6 | 4 |
| 9 | 0.3 | 0.6 | 0.25 | 0.72 | 0.02 | 0.64 | 1.45 | 0.02 | 0.73 | 1.97 | 0.02 | 5 | 1 | 6 | 2 |
| 9 | 0.3 | 1.0 | 0.51 | 1.86 | 0.00 | 0.91 | 2.84 | 0.00 | 1.06 | 2.17 | 0.00 | 5 | 2 | 6 | 3 |
| 9 | 0.6 | 0.3 | 0.64 | 2.06 | 0.16 | 1.02 | 3.59 | 0.02 | 0.82 | 2.22 | 0.02 | 5 | 2 | 6 | 3 |
| 9 | 0.6 | 0.6 | 1.30 | 1.89 | 0.34 | 1.83 | 3.63 | 0.88 | 2.13 | 3.31 | 1.08 | 5 | 1 | 6 | 2 |
| 9 | 0.6 | 1.0 | 2.66 | 4.77 | 1.09 | 3.64 | 9.99 | 0.61 | 3.40 | 5.84 | 0.75 | 2 | 1 | 3 | 2 |
| 9 | 0.9 | 0.3 | 1.48 | 3.44 | 0.58 | 1.58 | 7.61 | 0.22 | 1.49 | 2.95 | 0.19 | 2 | 1 | 3 | 2 |
| 9 | 0.9 | 0.6 | 1.62 | 2.70 | 0.45 | 1.80 | 4.61 | 0.02 | 2.59 | 3.86 | 0.02 | 4 | 1 | 5 | 2 |
| 9 | 0.9 | 1.0 | 5.52 | 11.67 | 0.83 | 4.17 | 15.30 | 0.80 | 4.82 | 7.80 | 1.88 | 2 | 1 | 3 | 2 |
| 10 | 0.3 | 0.3 | 0.73 | 1.31 | 0.41 | 1.22 | 3.24 | 0.03 | 1.06 | 2.27 | 0.02 | 5 | 1 | 6 | 2 |
| 10 | 0.3 | 0.6 | 1.68 | 3.17 | 0.52 | 2.64 | 5.69 | 0.56 | 1.90 | 3.88 | 0.41 | 4 | 1 | 5 | 2 |
| 10 | 0.3 | 1.0 | 3.43 | 15.25 | 0.36 | 5.38 | 29.17 | 0.53 | 3.45 | 6.83 | 1.23 | 5 | 1 | 6 | 2 |
| 10 | 0.6 | 0.3 | 1.64 | 3.05 | 0.61 | 2.57 | 5.45 | 0.56 | 2.11 | 3.55 | 0.70 | 4 | 1 | 5 | 2 |
| 10 | 0.6 | 0.6 | 2.83 | 5.14 | 0.95 | 3.51 | 8.80 | 1.36 | 3.64 | 5.95 | 2.30 | 5 | 1 | 6 | 2 |
| 10 | 0.6 | 1.0 | 7.30 | 13.42 | 2.33 | 16.11 | 57.00 | 2.84 | 7.34 | 13.70 | 2.38 | 5 | 1 | 6 | 2 |
| 10 | 0.9 | 0.3 | 3.01 | 6.00 | 1.36 | 3.69 | 6.88 | 2.00 | 3.08 | 5.30 | 0.73 | 3 | 1 | 4 | 2 |
| 10 | 0.9 | 0.6 | 4.82 | 11.19 | 1.36 | 9.21 | 23.52 | 1.34 | 7.13 | 15.99 | 3.81 | 3 | 1 | 4 | 2 |
| 10 | 0.9 | 1.0 | 9.50 | 26.55 | 1.49 | 18.46 | 99.11 | 1.05 | 10.22 | 18.31 | 3.64 | 5 | 1 | 6 | 2 |
| 12 | 0.3 | 0.3 | 4.96 | 13.28 | 1.05 | 9.49 | 30.17 | 0.33 | 4.38 | 8.39 | 0.66 | 7 | 6 | 5 | 2 |
| 12 | 0.3 | 0.6 | 5.20 | 12.02 | 1.74 | 7.87 | 27.02 | 1.72 | 5.75 | 10.31 | 2.69 | 2 | 1 | 3 | 2 |
| 12 | 0.3 | 1.0 | 10.70 | 46.19 | 1.38 | 11.06 | 32.94 | 0.83 | 7.31 | 19.45 | 0.27 | 7 | 2 | 5 | 3 |
| 12 | 0.6 | 0.3 | 6.09 | 31.64 | 0.63 | 6.59 | 35.80 | 0.16 | 4.75 | 15.88 | 0.14 | 7 | 3 | 5 | 4 |
| 12 | 0.6 | 0.6 | 20.50 | 94.92 | 5.30 | 39.88 | 176.64 | 2.73 | 14.98 | 38.16 | 5.52 | 7 | 6 | 3 | 2 |
| 12 | 0.6 | 1.0 | 46.36 | 210.11 | 5.69 | 53.39 | 227.53 | 2.63 | 21.06 | 39.81 | 5.73 | 7 | 1 | 4 | 2 |
| 12 | 0.9 | 0.3 | 6.93 | 14.97 | 1.17 | 15.10 | 53.22 | 3.49 | 5.63 | 15.83 | 1.67 | 7 | 6 | 3 | 2 |
| 12 | 0.9 | 0.6 | 13.46 | 24.74 | 5.67 | 23.43 | 50.16 | 7.09 | 13.32 | 23.45 | 6.03 | 7 | 1 | 6 | 2 |
| 12 | 0.9 | 1.0 | 25.58 | 135.33 | 3.08 | 40.88 | 233.05 | 1.45 | 20.37 | 67.38 | 0.50 | 7 | 1 | 3 | 2 |
| 15 | 0.3 | 0.3 | 9.88 | 20.19 | 3.23 | 21.76 | 62.50 | 7.84 | 10.46 | 24.88 | 5.41 | 5 | 1 | 6 | 2 |
| 15 | 0.3 | 0.6 | 41.40 | 78.84 | 8.34 | 70.03 | 133.81 | 19.99 | 21.98 | 36.78 | 10.81 | 7 | 6 | 6 | 5 |
| 15 | 0.3 | 1.0 | 85.98 | 233.27 | 10.58 | 116.60 | 339.63 | 12.08 | 51.40 | 121.55 | 7.24 | 7 | 1 | 6 | 2 |
| 15 | 0.6 | 0.3 | 33.19 | 125.38 | 5.24 | 67.22 | 237.17 | 13.78 | 13.50 | 42.16 | 5.53 | 7 | 6 | 3 | 4 |
| 15 | 0.6 | 0.6 | 39.67 | 92.64 | 8.33 | 71.46 | 225.02 | 18.56 | 23.51 | 39.86 | 7.38 | 7 | 1 | 5 | 2 |
| 15 | 0.6 | 1.0 | 228.90 | 1501.73 | 11.08 | 220.51 | 1295.81 | 11.84 | 69.84 | 271.49 | 14.23 | 7 | 1 | 5 | 2 |
| 15 | 0.9 | 0.3 | 70.80 | 199.75 | 12.84 | 77.54 | 296.66 | 8.92 | 34.75 | 85.03 | 4.69 | 7 | 3 | 5 | 4 |
| 15 | 0.9 | 0.6 | 226.52 | 1432.84 | 12.47 | 321.16 | 2252.91 | 9.59 | 142.73 | 951.64 | 14.17 | 7 | 1 | 6 | 2 |
| 15 | 0.9 | 1.0 | 197.03 | 644.44 | 37.45 | 242.47 | 754.94 | 27.75 | 78.47 | 232.17 | 23.45 | 7 | 3 | 6 | 4 |
| 20 | 0.3 | 0.3 | 850.45 | 3599.82 | 12.48 | 920.30 | 3599.72 | 7.05 | 212.77 | 716.43 | 4.64 | 7 | 3 | 5 | 4 |
| 20 | 0.3 | 0.6 | 2645.31 | 3600.04 | 130.50 | 2797.62 | 3599.99 | 427.50 | 2153.49 | 3599.88 | 61.52 | 7 | 6 | 4 | 3 |
| 20 | 0.3 | 1.0 | 1970.84 | 3599.84 | 73.52 | 2119.13 | 3600.12 | 145.16 | 1617.00 | 3599.95 | 48.31 | 7 | 1 | 4 | 2 |
| 20 | 0.6 | 0.3 | 981.69 | 3599.71 | 36.74 | 1490.85 | 3599.30 | 91.33 | 580.50 | 2170.87 | 39.00 | 7 | 6 | 4 | 2 |
| 20 | 0.6 | 0.6 | 2565.72 | 3600.02 | 73.25 | 2570.47 | 3599.91 | 123.61 | 2033.48 | 3599.65 | 32.53 | 7 | 1 | 4 | 2 |
| 20 | 0.6 | 1.0 | 3202.41 | 3600.08 | 1229.05 | 3365.06 | 3599.91 | 1255.95 | 3208.31 | 3599.89 | 357.49 | 2 | 1 | 3 | 2 |
| 20 | 0.9 | 0.3 | 2364.13 | 3599.84 | 98.53 | 2153.84 | 3599.54 | 115.53 | 1592.37 | 3599.37 | 52.99 | 7 | 1 | 5 | 2 |
| 20 | 0.9 | 0.6 | 3160.54 | 3599.99 | 492.68 | 2994.83 | 3599.83 | 307.46 | 2160.30 | 3599.96 | 200.57 | 7 | 3 | 3 | 4 |
| 20 | 0.9 | 1.0 | 3599.63 | 3600.06 | 3599.14 | 3500.59 | 3599.74 | 2610.90 | 3466.21 | 3600.05 | 2265.20 | 1, 7 | 3 | 2 | 4 |

off Xpress proprietary cuts. Finally, in the 7th version we used Xpress proprietary cuts in combination with valid inequalities (2)–(5). We used the following algorithms for separating inequalities (2)–(5):

*Separation oracle for the* 2*-partition inequalities*: We used a slightly modified version of the heuristic proposed by Grötschel and Wakabayashi [8] and Tcha et al. [11] to separate the 2-partition inequalities. Specifically, for every vertex $v \in V$ we consider the set of vertices $W := \{v' \in V \backslash \{v\} | w(v,v') > 0\}$. Then, we randomly pick two vertices, say $i$ and $j$, from $W$ and set $T=\{i,j\}$. Subsequently, for every vertex $k$ in $W-T$, we check if $w(k,v) - w(E(\{k\},T))$ is strictly positive and if so we set $T = T \cup \{k\}$. Finally, we check if $|T| \geq 3$ and $w(E(\{v\},T) - w(E(T)) > 1$. If both conditions hold, we add the violated inequality $w(E(\{v\},T) - w(E(T)) \leq 1$. Note that, if $|T| = 2$, the 2-partition inequality reduces to a triangle inequality, which is already present in the formulation.

*Separation oracle for the lower general clique inequalities*: As observed in Özsoy and Labbé [7], inequalities (3) are not facet defining for the SCGP polytope when $|Q| \leq \lfloor |V|/F_L \rfloor$. This fact can be exploited for separating the lower general clique inequalities. Specifically, for every $t \in \{\lfloor |V|/F_L \rfloor + 1, \lfloor |V|/F_L \rfloor + 2, \ldots, |V|\}$ we randomly pick a subset $Q$ of $V$ such that $|Q| = t$. We check whether $w(E(Q)) < (k-q)\binom{p}{2} + q\binom{p+1}{2}$ and if so, we add the inequality $w(E(Q)) \geq (k-q)\binom{p}{2} + q\binom{p+1}{2}$. We apply this procedure 100 times to every fractional solution encountered in the branch-and-cut tree.

*Separation oracle for the upper general clique inequalities*: As observed in Labbé and Özsoy [7], inequalities (4) are not facet defining for the SCGP polytope when $|Q| \leq \phi_1^U$ or $|Q| \geq |V| - \phi_{\bar{k}}^U$, being $\bar{k} = \lceil |V|/F_U \rceil$. This fact can be exploited for separating the upper general clique inequalities. Specifically, for every $t \in \{\phi_1^U + 1, \phi_1^U + 2, \ldots, |V| - \phi_{\bar{k}}^U - 1\}$, we randomly pick a subset $Q$ of $V$ such that $|Q| = t$ and we check whether $w(E(Q)) > \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2}$. If so we add the inequality

**Table 4**
Performances of the seven implementations of formulation (1) with respect to the gap (part 1).

| $\|V\|$ | Ed | Dd | Gap (%) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st cut set: Xpress cuts | | | 2nd cut set: valid ineq. (2) | | | 3rd cut set: valid ineq. (3) | | | 4th cut set: valid ineq. (4) | | |
| | | | Average | Max | Min | Average | Max | Min | Average | Max | Min | Average | Max | Min |
| 9 | 0.3 | 0.3 | 7.68 | 21.31 | 0.56 | 7.55 | 21.28 | 2.43 | 7.60 | 20.92 | 3.68 | 7.60 | 20.92 | 3.68 |
| 9 | 0.3 | 0.6 | 3.47 | 13.73 | 0.46 | 3.34 | 13.60 | 5.10 | 3.44 | 13.73 | 5.88 | 3.44 | 13.73 | 5.88 |
| 9 | 0.3 | 1.0 | 7.77 | 16.02 | 0.22 | 7.34 | 15.66 | 0.01 | 7.75 | 16.02 | 0.22 | 7.75 | 16.02 | 0.22 |
| 9 | 0.6 | 0.3 | 5.57 | 16.75 | 0.84 | 5.51 | 16.50 | 0.80 | 5.57 | 16.75 | 0.84 | 5.57 | 16.75 | 0.84 |
| 9 | 0.6 | 0.6 | 12.87 | 24.79 | 3.57 | 12.78 | 24.76 | 3.57 | 12.86 | 24.63 | 3.57 | 12.86 | 24.63 | 3.57 |
| 9 | 0.6 | 1.0 | 12.14 | 26.14 | 2.12 | 11.83 | 25.87 | 2.12 | 12.14 | 26.14 | 2.12 | 12.14 | 26.14 | 2.12 |
| 9 | 0.9 | 0.3 | 9.14 | 16.27 | 1.34 | 9.10 | 16.15 | 1.34 | 9.14 | 16.27 | 1.34 | 9.14 | 16.27 | 1.34 |
| 9 | 0.9 | 0.6 | 9.37 | 16.42 | 5.02 | 9.25 | 16.42 | 4.88 | 9.36 | 16.42 | 5.02 | 9.36 | 16.42 | 5.02 |
| 9 | 0.9 | 1.0 | 10.32 | 17.45 | 4.28 | 10.05 | 17.45 | 4.10 | 10.32 | 17.45 | 4.28 | 10.32 | 17.45 | 4.28 |
| 10 | 0.3 | 0.3 | 7.63 | 23.80 | 2.11 | 7.63 | 23.80 | 0.54 | 7.63 | 23.80 | 2.11 | 7.63 | 23.80 | 2.11 |
| 10 | 0.3 | 0.6 | 11.54 | 31.37 | 0.43 | 11.51 | 31.37 | 0.43 | 11.54 | 31.37 | 0.43 | 11.54 | 31.37 | 0.43 |
| 10 | 0.3 | 1.0 | 11.43 | 21.92 | 3.08 | 11.37 | 21.92 | 3.08 | 11.43 | 21.92 | 3.08 | 11.43 | 21.92 | 3.08 |
| 10 | 0.6 | 0.3 | 7.28 | 14.68 | 3.33 | 7.25 | 14.61 | 3.33 | 7.28 | 14.68 | 3.33 | 7.28 | 14.68 | 3.33 |
| 10 | 0.6 | 0.6 | 8.89 | 20.45 | 3.20 | 8.86 | 20.45 | 3.15 | 8.89 | 20.45 | 3.18 | 8.89 | 20.45 | 3.18 |
| 10 | 0.6 | 1.0 | 13.00 | 18.59 | 7.38 | 12.85 | 18.56 | 7.38 | 12.98 | 18.59 | 7.38 | 12.98 | 18.59 | 7.38 |
| 10 | 0.9 | 0.3 | 8.57 | 15.89 | 0.14 | 8.55 | 15.80 | 0.14 | 8.56 | 15.89 | 0.14 | 8.56 | 15.89 | 0.14 |
| 10 | 0.9 | 0.6 | 10.59 | 18.03 | 3.25 | 10.48 | 17.91 | 3.01 | 10.58 | 18.03 | 3.25 | 10.58 | 18.03 | 3.25 |
| 10 | 0.9 | 1.0 | 11.44 | 19.52 | 1.61 | 11.25 | 19.08 | 1.55 | 11.43 | 19.52 | 1.61 | 11.43 | 19.52 | 1.61 |
| 12 | 0.3 | 0.3 | 7.77 | 14.97 | 0.75 | 7.66 | 14.70 | 0.75 | 7.77 | 14.97 | 0.75 | 7.77 | 14.97 | 0.75 |
| 12 | 0.3 | 0.6 | 10.27 | 21.61 | 4.22 | 10.06 | 21.47 | 4.06 | 10.27 | 21.61 | 4.21 | 10.27 | 21.61 | 4.21 |
| 12 | 0.3 | 1.0 | 9.43 | 19.69 | 3.10 | 9.33 | 19.61 | 3.10 | 9.43 | 19.69 | 0.89 | 9.43 | 19.69 | 0.01 |
| 12 | 0.6 | 0.3 | 6.11 | 15.17 | 0.30 | 6.08 | 14.99 | 0.25 | 6.11 | 15.17 | 0.30 | 6.11 | 15.17 | 0.30 |
| 12 | 0.6 | 0.6 | 10.95 | 19.82 | 3.58 | 10.81 | 19.53 | 3.57 | 10.95 | 19.82 | 3.58 | 10.95 | 19.82 | 3.58 |
| 12 | 0.6 | 1.0 | 9.71 | 15.77 | 2.42 | 9.52 | 15.51 | 2.39 | 9.71 | 15.77 | 2.42 | 9.71 | 15.77 | 2.42 |
| 12 | 0.9 | 0.3 | 7.69 | 15.41 | 1.79 | 7.65 | 15.33 | 1.77 | 7.69 | 15.41 | 1.79 | 7.69 | 15.41 | 1.79 |
| 12 | 0.9 | 0.6 | 9.50 | 16.03 | 2.24 | 9.34 | 15.98 | 2.14 | 9.45 | 16.03 | 1.78 | 9.45 | 16.03 | 1.78 |
| 12 | 0.9 | 1.0 | 8.62 | 16.34 | 0.03 | 8.47 | 16.26 | 0.03 | 8.62 | 16.34 | 0.03 | 8.62 | 16.34 | 0.03 |
| 15 | 0.3 | 0.3 | 4.75 | 9.53 | 1.55 | 4.71 | 9.48 | 1.55 | 4.75 | 9.53 | 1.55 | 4.75 | 9.53 | 1.55 |
| 15 | 0.3 | 0.6 | 7.31 | 10.48 | 2.70 | 7.21 | 10.33 | 2.52 | 7.31 | 10.48 | 2.70 | 7.31 | 10.48 | 2.70 |
| 15 | 0.3 | 1.0 | 9.17 | 13.11 | 3.52 | 9.06 | 13.01 | 3.52 | 9.17 | 13.11 | 3.52 | 9.17 | 13.11 | 3.52 |
| 15 | 0.6 | 0.3 | 5.13 | 10.09 | 2.24 | 5.10 | 9.93 | 2.24 | 5.13 | 10.09 | 2.24 | 5.13 | 10.09 | 2.24 |
| 15 | 0.6 | 0.6 | 5.36 | 8.88 | 1.51 | 5.28 | 8.79 | 1.51 | 5.36 | 8.88 | 1.51 | 5.36 | 8.88 | 1.51 |
| 15 | 0.6 | 1.0 | 7.63 | 13.53 | 2.31 | 7.48 | 13.41 | 2.23 | 7.63 | 13.53 | 2.31 | 7.63 | 13.53 | 2.31 |
| 15 | 0.9 | 0.3 | 6.56 | 9.11 | 0.61 | 6.52 | 9.10 | 0.61 | 6.56 | 9.11 | 0.61 | 6.56 | 9.11 | 0.61 |
| 15 | 0.9 | 0.6 | 8.66 | 13.70 | 2.17 | 8.44 | 13.53 | 2.07 | 8.66 | 13.70 | 2.17 | 8.66 | 13.70 | 2.17 |
| 15 | 0.9 | 1.0 | 7.89 | 11.84 | 3.31 | 7.77 | 11.66 | 3.31 | 7.89 | 11.84 | 3.31 | 7.89 | 11.84 | 3.31 |
| 20 | 0.3 | 0.3 | 8.88 | 13.75 | 0.63 | 8.54 | 13.14 | 0.52 | 9.19 | 15.73 | 0.63 | 9.12 | 15.01 | 0.63 |
| 20 | 0.3 | 0.6 | 15.80 | 24.33 | 7.63 | 14.83 | 20.19 | 7.44 | 14.69 | 19.60 | 7.63 | 15.39 | 21.02 | 7.63 |
| 20 | 0.3 | 1.0 | 11.76 | 18.67 | 5.70 | 10.71 | 15.85 | 5.54 | 10.43 | 16.08 | 5.70 | 11.21 | 19.12 | 5.70 |
| 20 | 0.6 | 0.3 | 10.64 | 16.77 | 5.92 | 10.10 | 15.40 | 5.56 | 10.70 | 17.99 | 5.92 | 10.83 | 19.27 | 5.92 |
| 20 | 0.6 | 0.6 | 15.98 | 29.66 | 3.92 | 13.69 | 26.37 | 3.91 | 14.53 | 26.69 | 3.92 | 14.89 | 26.69 | 3.92 |
| 20 | 0.6 | 1.0 | 18.55 | 32.91 | 6.26 | 16.31 | 23.98 | 6.11 | 17.96 | 28.60 | 6.26 | 17.61 | 26.64 | 6.26 |
| 20 | 0.9 | 0.3 | 14.10 | 23.91 | 4.04 | 12.57 | 20.95 | 3.78 | 13.00 | 22.10 | 4.04 | 12.69 | 21.23 | 4.04 |
| 20 | 0.9 | 0.6 | 13.89 | 21.67 | 7.81 | 13.06 | 20.05 | 7.22 | 14.44 | 22.56 | 7.81 | 14.08 | 21.32 | 7.81 |
| 20 | 0.9 | 1.0 | 21.86 | 37.01 | 6.04 | 18.44 | 27.86 | 5.82 | 18.68 | 32.04 | 7.70 | 17.36 | 30.10 | 6.04 |

$w(E(Q)) \le \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2}$. We apply this procedure 100 times to every fractional solution encountered in the branch-and-cut tree. The whole procedure can be accelerated by precomputing the values $\{\phi_i^U\}$, $i=1,\ldots,\overline{k}$.

*Separation oracle for the cycle inequalities*: The cycle inequalities are separated by means of a heuristic algorithm working as follows. For every vertex $v \in V$ we set $V_c := \{v\}$, and we add the remaining vertices to $V_c$ by picking at random $(\phi_1^U - 1)$ vertices in $V \backslash \{v\}$. Subsequently, we check whether $w(E_c) > \phi_1^U - 2$. If so we add the inequality $w(E_c) \le \phi_1^U - 2$.

### 4.1. Discussion

Fig. 3 provides a general overview of the numerical results performed on the considered 450 random instances of the PHLRP. As general trend, implementation 7 performed the best resulting, in average, faster and tighter than any other implementation. Tables 2–7 show a more detailed and systematic analysis of the results. Specifically, Tables 2 and 3 show the performances of the seven implementations with respect to the solution time. Tables 4 and 5 show the performances of the implementations with respect to the gap, i.e., the difference between the optimal value found (for a specific instance) and the value of linear relaxation at root node of the branch-and-cut tree, divided by the optimal value. Finally, Tables 6 and 7 show the performances of the implementations with respect to the number of nodes explored in the branch-and-cut tree.

Each row of Tables 2 and 3 shows, for each implementation, the average, the maximum and the minimum solution time required to solve the corresponding 10 random instances. When the running time overcomes the time limit, it appears as around 3600 s in the tables. The 13th and 14th columns of Table 3 display the best and the worst performing implementations with respect to the average solution times. The columns show that, as size gets larger, implementation 7 performs generally better by resulting

**Table 5**
Performances of the seven implementations of formulation (1) with respect to the gap (part 2).

| \|V\| | Ed | Dd | Gap (%) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 5th cut set: valid ineq. (5) | | | 6th cut set: valid ineq. (2)–(5) | | | 7th cut set: Xpress Cuts+V.I. (2)–(5) | | | Among 1st–6th cut sets | | Among 2nd–5th cut sets | |
| | | | Average | Max | Min | Average | Max | Min | Average | Max | Min | Best | Worst | Best | Worst |
| 9 | 0.3 | 0.3 | 6.60 | 31.35 | 10.57 | 6.81 | 21.19 | 1.58 | 6.28 | 20.53 | 1.37 | 5 | 1 | 5 | 3, 4 |
| 9 | 0.3 | 0.6 | 6.52 | 30.72 | 9.13 | 2.48 | 13.35 | 4.83 | 1.65 | 12.80 | 4.36 | 6 | 5 | 2 | 5 |
| 9 | 0.3 | 1.0 | 6.85 | 33.09 | 10.08 | 7.23 | 15.52 | 0.55 | 6.54 | 15.02 | 0.11 | 5 | 1 | 5 | 3, 4 |
| 9 | 0.6 | 0.3 | 14.13 | 23.35 | 0.84 | 5.13 | 16.26 | 0.98 | 4.64 | 15.43 | 0.56 | 6 | 5 | 2 | 5 |
| 9 | 0.6 | 0.6 | 26.59 | 39.21 | 11.26 | 11.90 | 24.57 | 2.75 | 11.56 | 24.48 | 2.66 | 6 | 5 | 2 | 5 |
| 9 | 0.6 | 1.0 | 19.93 | 34.22 | 15.11 | 11.39 | 25.32 | 1.90 | 11.08 | 24.36 | 1.17 | 6 | 5 | 2 | 5 |
| 9 | 0.9 | 0.3 | 15.94 | 25.68 | 4.64 | 8.54 | 15.60 | 0.64 | 8.51 | 14.93 | 0.78 | 6 | 5 | 2 | 5 |
| 9 | 0.9 | 0.6 | 19.13 | 34.44 | 3.86 | 8.57 | 15.68 | 4.56 | 7.61 | 15.26 | 4.11 | 6 | 5 | 2 | 5 |
| 9 | 0.9 | 1.0 | 20.52 | 35.11 | 5.30 | 9.36 | 16.46 | 3.45 | 9.24 | 16.22 | 3.45 | 6 | 5 | 2 | 5 |
| 10 | 0.3 | 0.3 | 9.32 | 19.94 | 2.11 | 7.57 | 22.95 | 0.64 | 7.55 | 22.48 | 0.39 | 6 | 5 | 3, 4 | 5 |
| 10 | 0.3 | 0.6 | 12.35 | 30.89 | 2.76 | 11.11 | 30.71 | 0.66 | 10.91 | 29.97 | 0.76 | 6 | 5 | 2 | 5 |
| 10 | 0.3 | 1.0 | 11.27 | 21.92 | 2.52 | 11.12 | 21.31 | 2.45 | 11.01 | 20.74 | 1.66 | 6 | 1, 3, 4 | 5 | 3, 4 |
| 10 | 0.6 | 0.3 | 7.19 | 14.48 | 3.33 | 6.58 | 14.07 | 2.90 | 6.11 | 13.27 | 1.98 | 6 | 1, 3, 4 | 5 | 3, 4 |
| 10 | 0.6 | 0.6 | 8.89 | 20.04 | 3.18 | 8.30 | 19.84 | 2.46 | 7.67 | 19.09 | 1.57 | 6 | 1 | 2 | 5 |
| 10 | 0.6 | 1.0 | 14.00 | 18.58 | 7.38 | 12.52 | 18.09 | 7.28 | 11.81 | 17.74 | 6.91 | 6 | 5 | 2 | 5 |
| 10 | 0.9 | 0.3 | 9.09 | 15.89 | 0.14 | 8.05 | 15.12 | 0.54 | 7.40 | 14.42 | 0.39 | 6 | 5 | 2 | 5 |
| 10 | 0.9 | 0.6 | 11.54 | 17.86 | 3.20 | 9.77 | 17.52 | 2.89 | 9.01 | 17.32 | 2.87 | 6 | 5 | 2 | 5 |
| 10 | 0.9 | 1.0 | 11.79 | 19.50 | 0.89 | 10.70 | 18.67 | 0.80 | 9.82 | 18.16 | 0.79 | 6 | 5 | 2 | 5 |
| 12 | 0.3 | 0.3 | 8.94 | 14.97 | 2.22 | 6.76 | 14.63 | 0.71 | 6.02 | 14.60 | 0.40 | 6 | 5 | 2 | 5 |
| 12 | 0.3 | 0.6 | 10.59 | 21.60 | 4.19 | 9.29 | 21.28 | 3.14 | 8.60 | 20.95 | 2.79 | 6 | 5 | 2 | 5 |
| 12 | 0.3 | 1.0 | 10.18 | 19.69 | 3.10 | 8.69 | 19.30 | 3.01 | 8.12 | 18.90 | 2.58 | 6 | 5 | 2 | 5 |
| 12 | 0.6 | 0.3 | 6.76 | 15.17 | 0.28 | 5.69 | 14.26 | 0.00 | 5.36 | 13.50 | 1.00 | 6 | 5 | 2 | 5 |
| 12 | 0.6 | 0.6 | 11.39 | 19.82 | 3.57 | 10.39 | 18.87 | 3.27 | 10.09 | 17.92 | 2.66 | 6 | 5 | 2 | 5 |
| 12 | 0.6 | 1.0 | 10.41 | 15.77 | 3.99 | 9.38 | 14.73 | 2.13 | 9.11 | 14.60 | 2.05 | 6 | 5 | 2 | 5 |
| 12 | 0.9 | 0.3 | 8.25 | 15.41 | 1.74 | 6.99 | 15.09 | 1.12 | 6.01 | 14.82 | 1.04 | 6 | 5 | 2 | 5 |
| 12 | 0.9 | 0.6 | 9.81 | 16.03 | 2.10 | 9.30 | 15.01 | 1.33 | 8.68 | 14.69 | 0.36 | 6 | 5 | 2 | 5 |
| 12 | 0.9 | 1.0 | 9.18 | 16.34 | 4.29 | 8.30 | 15.85 | 0.23 | 7.65 | 15.57 | 0.65 | 6 | 5 | 2 | 5 |
| 15 | 0.3 | 0.3 | 4.75 | 9.53 | 1.55 | 4.02 | 8.64 | 1.54 | 3.52 | 8.25 | 1.45 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.3 | 0.6 | 7.31 | 10.48 | 2.70 | 6.92 | 10.13 | 1.89 | 6.57 | 9.90 | 1.61 | 6 | 1, 3, 4 | 2 | 3, 4 |
| 15 | 0.3 | 1.0 | 9.17 | 13.11 | 3.52 | 8.65 | 12.41 | 3.34 | 7.80 | 11.96 | 2.64 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.6 | 0.3 | 5.13 | 10.09 | 2.24 | 4.37 | 9.01 | 1.96 | 3.66 | 8.95 | 1.87 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.6 | 0.6 | 5.36 | 8.88 | 1.51 | 5.03 | 8.54 | 0.76 | 4.17 | 8.39 | 0.30 | 6 | 1, 3, 4 | 2 | 3, 4 |
| 15 | 0.6 | 1.0 | 7.63 | 13.53 | 2.31 | 6.99 | 12.69 | 2.15 | 6.90 | 12.37 | 1.30 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.9 | 0.3 | 6.56 | 9.11 | 0.61 | 5.61 | 8.35 | 0.96 | 5.55 | 7.49 | 0.91 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.9 | 0.6 | 8.66 | 13.70 | 2.17 | 7.73 | 13.25 | 1.79 | 6.84 | 13.17 | 1.20 | 6 | 1, 3, 4 | 2 | 3, 4 |
| 15 | 0.9 | 1.0 | 7.89 | 11.84 | 3.31 | 6.78 | 10.86 | 3.03 | 6.73 | 10.34 | 2.63 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 20 | 0.3 | 0.3 | 9.58 | 15.01 | 5.30 | 7.96 | 12.43 | 0.48 | 7.81 | 11.76 | 0.28 | 6 | 5 | 2 | 5 |
| 20 | 0.3 | 0.6 | 14.78 | 19.60 | 7.63 | 13.85 | 19.92 | 6.78 | 13.26 | 19.53 | 5.78 | 6 | 1 | 3 | 4 |
| 20 | 0.3 | 1.0 | 11.21 | 19.12 | 5.70 | 10.54 | 14.91 | 5.15 | 9.61 | 13.92 | 4.81 | 3 | 1 | 3 | 4, 5 |
| 20 | 0.6 | 0.3 | 10.83 | 19.27 | 5.92 | 9.75 | 15.23 | 5.31 | 8.94 | 14.73 | 5.28 | 6 | 4, 5 | 2 | 4, 5 |
| 20 | 0.6 | 0.6 | 14.89 | 26.69 | 3.92 | 12.74 | 26.28 | 2.95 | 12.56 | 25.41 | 2.54 | 6 | 1 | 2 | 4, 5 |
| 20 | 0.6 | 1.0 | 17.61 | 26.64 | 6.26 | 15.96 | 23.04 | 5.45 | 15.15 | 22.86 | 5.19 | 6 | 1 | 2 | 3 |
| 20 | 0.9 | 0.3 | 13.08 | 22.10 | 4.04 | 12.14 | 20.92 | 3.45 | 11.65 | 20.55 | 2.54 | 6 | 1 | 2 | 5 |
| 20 | 0.9 | 0.6 | 13.97 | 21.32 | 7.81 | 13.03 | 19.77 | 7.02 | 12.03 | 19.47 | 6.16 | 6 | 3 | 2 | 3 |
| 20 | 0.9 | 1.0 | 18.92 | 34.98 | 8.01 | 18.02 | 27.38 | 5.76 | 17.35 | 27.09 | 5.64 | 4 | 1 | 4 | 5 |

**Table 6**
Performances of the seven implementations of formulation (1) with respect to the nodes explored in the tree search (part 1).

| $\|V\|$ | Ed | Dd | Number of nodes in the branch-and-cut tree | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st cut set: Xpress cuts | | | 2nd cut set: valid ineq. (2) | | | 3rd cut set: valid ineq. (3) | | | 4th cut set: valid ineq. (4) | | |
| | | | Average | Max | Min | Average | Max | Min | Average | Max | Min | Average | Max | Min |
| 9 | 0.3 | 0.3 | 4.2 | 19 | 1 | 4.2 | 21 | 1 | 5.4 | 17 | 1 | 6.4 | 15 | 1 |
| 9 | 0.3 | 0.6 | 8.1 | 27 | 1 | 8.1 | 21 | 1 | 7.5 | 25 | 1 | 7.7 | 25 | 1 |
| 9 | 0.3 | 1.0 | 7.7 | 27 | 1 | 7.7 | 29 | 1 | 9.7 | 27 | 1 | 9.1 | 27 | 1 |
| 9 | 0.6 | 0.3 | 8.9 | 31 | 1 | 8.9 | 57 | 1 | 11.8 | 35 | 1 | 11.8 | 35 | 1 |
| 9 | 0.6 | 0.6 | 28.8 | 131 | 1 | 28.8 | 95 | 7 | 29.6 | 81 | 5 | 28.2 | 81 | 5 |
| 9 | 0.6 | 1.0 | 42.9 | 187 | 1 | 42.9 | 139 | 1 | 36.8 | 115 | 1 | 42.4 | 145 | 1 |
| 9 | 0.9 | 0.3 | 38.2 | 171 | 1 | 38.2 | 147 | 1 | 39.0 | 199 | 1 | 39.0 | 199 | 1 |
| 9 | 0.9 | 0.6 | 16.2 | 39 | 1 | 16.2 | 71 | 1 | 19.8 | 51 | 1 | 20.0 | 51 | 1 |
| 9 | 0.9 | 1.0 | 39.4 | 215 | 1 | 39.4 | 167 | 1 | 42.4 | 195 | 1 | 42.4 | 195 | 1 |
| 10 | 0.3 | 0.3 | 33.5 | 153 | 1 | 33.5 | 203 | 1 | 30.3 | 81 | 1 | 29.9 | 81 | 1 |
| 10 | 0.3 | 0.6 | 50.4 | 147 | 1 | 50.4 | 149 | 5 | 60.0 | 199 | 9 | 63.6 | 199 | 7 |
| 10 | 0.3 | 1.0 | 69.8 | 271 | 9 | 69.8 | 229 | 7 | 79.8 | 441 | 5 | 79.8 | 441 | 5 |
| 10 | 0.6 | 0.3 | 48.2 | 127 | 9 | 48.2 | 77 | 9 | 49.0 | 119 | 13 | 49.6 | 119 | 13 |
| 10 | 0.6 | 0.6 | 64.4 | 255 | 11 | 64.4 | 289 | 13 | 84.1 | 388 | 11 | 84.1 | 388 | 11 |
| 10 | 0.6 | 1.0 | 143.0 | 283 | 25 | 143.0 | 596 | 19 | 175.2 | 764 | 17 | 175.2 | 764 | 17 |
| 10 | 0.9 | 0.3 | 78.2 | 365 | 17 | 78.2 | 220 | 11 | 112.8 | 382 | 11 | 112.8 | 382 | 11 |
| 10 | 0.9 | 0.6 | 179.2 | 827 | 9 | 119.2 | 549 | 9 | 123.3 | 352 | 11 | 123.3 | 352 | 11 |
| 10 | 0.9 | 1.0 | 196.5 | 651 | 1 | 196.5 | 719 | 1 | 201.2 | 959 | 1 | 202.8 | 959 | 1 |
| 12 | 0.3 | 0.3 | 90.3 | 249 | 1 | 90.3 | 503 | 1 | 147.8 | 427 | 1 | 148.4 | 427 | 1 |
| 12 | 0.3 | 0.6 | 71.2 | 249 | 15 | 61.2 | 241 | 13 | 84.4 | 257 | 9 | 84.4 | 257 | 9 |
| 12 | 0.3 | 1.0 | 96.1 | 511 | 1 | 96.1 | 495 | 2 | 105.5 | 605 | 2 | 105.5 | 605 | 2 |
| 12 | 0.6 | 0.3 | 165.5 | 1232 | 1 | 165.5 | 697 | 1 | 189.4 | 1299 | 1 | 189.4 | 1299 | 1 |
| 12 | 0.6 | 0.6 | 315.6 | 971 | 17 | 315.6 | 961 | 15 | 356.2 | 1334 | 13 | 356.2 | 1334 | 13 |
| 12 | 0.6 | 1.0 | 242.9 | 681 | 11 | 242.9 | 911 | 11 | 316.7 | 1519 | 7 | 316.7 | 1519 | 7 |
| 12 | 0.9 | 0.3 | 137.0 | 553 | 13 | 137.0 | 763 | 19 | 158.0 | 529 | 10 | 158.9 | 529 | 19 |
| 12 | 0.9 | 0.6 | 141.0 | 291 | 9 | 141.0 | 349 | 3 | 136.0 | 321 | 3 | 136.0 | 321 | 3 |
| 12 | 0.9 | 1.0 | 188.3 | 1041 | 1 | 188.3 | 903 | 1 | 172.9 | 1210 | 1 | 172.9 | 1210 | 1 |
| 15 | 0.3 | 0.3 | 174.4 | 649 | 15 | 174.4 | 439 | 19 | 123.3 | 516 | 29 | 123.3 | 516 | 29 |
| 15 | 0.3 | 0.6 | 229.9 | 453 | 13 | 229.9 | 902 | 19 | 292.9 | 677 | 21 | 292.9 | 677 | 21 |
| 15 | 0.3 | 1.0 | 679.5 | 1735 | 31 | 379.5 | 1484 | 21 | 355.1 | 999 | 18 | 355.1 | 999 | 18 |
| 15 | 0.6 | 0.3 | 192.5 | 1054 | 19 | 192.5 | 858 | 33 | 459.6 | 1328 | 35 | 455.0 | 1328 | 35 |
| 15 | 0.6 | 0.6 | 216.7 | 775 | 27 | 216.7 | 485 | 19 | 222.3 | 803 | 25 | 222.3 | 803 | 25 |
| 15 | 0.6 | 1.0 | 575.0 | 2465 | 43 | 475.0 | 1939 | 25 | 519.7 | 2987 | 27 | 519.7 | 2987 | 27 |
| 15 | 0.9 | 0.3 | 583.6 | 1579 | 5 | 583.6 | 1724 | 9 | 620.1 | 1373 | 19 | 620.1 | 1373 | 19 |
| 15 | 0.9 | 0.6 | 1476.6 | 11155 | 17 | 1450.2 | 10569 | 15 | 1109.5 | 7917 | 9 | 1109.7 | 7917 | 9 |
| 15 | 0.9 | 1.0 | 441.3 | 1528 | 27 | 441.3 | 1149 | 23 | 524.8 | 1675 | 39 | 524.8 | 1675 | 39 |
| 20 | 0.3 | 0.3 | 2097.4 | 5901 | 3 | 1105.0 | 4633 | 1 | 1615.6 | 5043 | 7 | 1715.9 | 5839 | 7 |
| 20 | 0.3 | 0.6 | 2704.3 | 3714 | 395 | 1089.9 | 5480 | 361 | 2475.2 | 4718 | 439 | 2391.9 | 3282 | 439 |
| 20 | 0.3 | 1.0 | 946.2 | 1688 | 49 | 1497.3 | 2845 | 29 | 1638.1 | 3484 | 27 | 1011.5 | 1991 | 27 |
| 20 | 0.6 | 0.3 | 3186.0 | 7523 | 133 | 4422.0 | 5440 | 97 | 2812.8 | 11927 | 63 | 2188.1 | 5680 | 63 |
| 20 | 0.6 | 0.6 | 2084.1 | 5571 | 51 | 2264.3 | 2782 | 63 | 2023.3 | 3560 | 57 | 1889.6 | 4357 | 57 |
| 20 | 0.6 | 1.0 | 1127.2 | 2064 | 525 | 1879.6 | 1426 | 173 | 1029.5 | 1508 | 620 | 1135.4 | 1510 | 620 |
| 20 | 0.9 | 0.3 | 3082.0 | 6722 | 141 | 1999.1 | 5391 | 242 | 3120.3 | 8110 | 107 | 4093.9 | 8136 | 107 |
| 20 | 0.9 | 0.6 | 2103.4 | 3358 | 301 | 2548.5 | 2559 | 627 | 1808.7 | 2967 | 317 | 2289.5 | 3520 | 317 |
| 20 | 0.9 | 1.0 | 1094.8 | 1697 | 548 | 1094.8 | 1903 | 927 | 948.7 | 1305 | 686 | 1820.6 | 2341 | 1581 |

significantly faster than the others. On the contrary, the exclusive use of Xpress proprietary cuts generally leads to poor performances. This fact implicitly evidences the efficacy of the valid inequalities discussed in Section 3.

Implementation 2 turns out to be generally the worst in terms of solution time. This fact is in contrast with Tables 4 and 5 which show the ability of implementation 2 to provide tight lower bounds for the problem. This behavior is possibly due to the hardness of separating inequalities (2), so that the time spent for the separation is not worth its use. On the other hand, we can observe from the penultimate column that implementation 6 performs the best in a considerable number of cases. Implementations 3 and 5 also perform very well and yield quite good solution times when compared with the other implementations under consideration. In the absence of Xpress proprietary cuts we would recommend the use of valid inequalities (2)–(5) jointly in a branch-and-cut framework.

Tables 4 and 5 show the effectiveness of the valid inequalities in providing tight lower bounds for the problem. Each row displays, for each implementation, the average, the maximum, and the minimum gap on the corresponding 10 random instances. If for some instance the running time exceeded the limit time, the best so far primal bound is used instead of the optimal value.

We observed that implementation 7 performs generally better by always yielding to the best average gaps. When excluding implementation 7 from the analysis we observed that implementation 6 generally performs the best and globally better than implementation 1 (see the 13th and 14th columns of Table 5). This fact confirms again that the joint use of valid inequalities (2)–(5) significantly improves over Xpress proprietary cuts in terms of LP relaxations. As implementation 6 employs all the corresponding valid inequalities there is no surprise in seeing that it performs better than implementations 2–5. Interestingly, due to

**Table 7**
Performances of the seven implementations of formulation (1) with respect to the nodes explored in the tree search (part 2).

| $\|V\|$ | Ed | Dd | Number of nodes in the branch-and-cut tree | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 5th cut set: valid ineq. (5) | | | 6th cut set: valid ineq. (2)–(5) | | | 7th cut set: Xpress Cuts+V.I. (2)–(5) | | | Best cut set | Worst cut set |
| | | | Average | Max | Min | Average | Max | Min | Average | Max | Min | | |
| 9 | 0.3 | 0.3 | 8.5 | 21 | 1 | 5.6 | 13 | 1 | 6.2 | 19 | 1 | 1, 2 | 5 |
| 9 | 0.3 | 0.6 | 9.6 | 51 | 1 | 7.8 | 25 | 1 | 5.8 | 27 | 1 | 7 | 5 |
| 9 | 0.3 | 1.0 | 18.7 | 103 | 1 | 9.7 | 27 | 1 | 10.1 | 27 | 1 | 1, 2 | 5 |
| 9 | 0.6 | 0.3 | 40.6 | 173 | 5 | 11.8 | 35 | 1 | 15.2 | 31 | 1 | 1, 2 | 5 |
| 9 | 0.6 | 0.6 | 63.4 | 103 | 15 | 29.0 | 81 | 5 | 29.8 | 131 | 1 | 4 | 5 |
| 9 | 0.6 | 1.0 | 60.0 | 145 | 15 | 39.6 | 117 | 1 | 43.4 | 187 | 1 | 3 | 5 |
| 9 | 0.9 | 0.3 | 57.9 | 179 | 11 | 38.2 | 199 | 1 | 32.2 | 171 | 1 | 7 | 5 |
| 9 | 0.9 | 0.6 | 36.4 | 79 | 1 | 19.8 | 51 | 1 | 21.7 | 39 | 1 | 1, 2 | 5 |
| 9 | 0.9 | 1.0 | 103.8 | 235 | 7 | 42.4 | 195 | 1 | 38.4 | 215 | 1 | 7 | 5 |
| 10 | 0.3 | 0.3 | 52.1 | 205 | 8 | 32.5 | 81 | 1 | 46.2 | 153 | 1 | 4 | 5 |
| 10 | 0.3 | 0.6 | 71.0 | 221 | 3 | 56.4 | 199 | 5 | 42.2 | 147 | 1 | 7 | 5 |
| 10 | 0.3 | 1.0 | 67.8 | 343 | 1 | 80.0 | 441 | 7 | 62.6 | 271 | 9 | 7 | 6 |
| 10 | 0.6 | 0.3 | 43.2 | 119 | 13 | 48.2 | 119 | 13 | 37.8 | 127 | 9 | 7 | 4 |
| 10 | 0.6 | 0.6 | 72.7 | 269 | 8 | 87.3 | 388 | 11 | 79.4 | 255 | 11 | 1, 2 | 6 |
| 10 | 0.6 | 1.0 | 146.1 | 391 | 15 | 176.0 | 764 | 17 | 167.5 | 283 | 25 | 1, 2 | 6 |
| 10 | 0.9 | 0.3 | 97.6 | 382 | 13 | 110.8 | 382 | 11 | 112.8 | 365 | 17 | 1, 2 | 3, 4, 7 |
| 10 | 0.9 | 0.6 | 144.9 | 441 | 7 | 130.9 | 352 | 11 | 162.3 | 827 | 9 | 2 | 1 |
| 10 | 0.9 | 1.0 | 170.4 | 769 | 1 | 201.8 | 959 | 1 | 195.9 | 651 | 1 | 5 | 4 |
| 12 | 0.3 | 0.3 | 172.9 | 457 | 9 | 147.6 | 427 | 1 | 140.5 | 249 | 1 | 1, 2 | 5 |
| 12 | 0.3 | 0.6 | 71.6 | 209 | 9 | 85.2 | 257 | 9 | 68.0 | 249 | 15 | 2 | 6 |
| 12 | 0.3 | 1.0 | 133.1 | 605 | 2 | 105.5 | 605 | 2 | 126.7 | 511 | 1 | 1, 2 | 5 |
| 12 | 0.6 | 0.3 | 205.7 | 1299 | 1 | 189.6 | 1299 | 1 | 159.6 | 1232 | 1 | 7 | 5 |
| 12 | 0.6 | 0.6 | 348.0 | 1334 | 19 | 341.4 | 1334 | 13 | 327.2 | 971 | 17 | 1, 2 | 3, 4 |
| 12 | 0.6 | 1.0 | 325.5 | 1519 | 15 | 316.7 | 1519 | 7 | 387.0 | 681 | 11 | 1, 2 | 7 |
| 12 | 0.9 | 0.3 | 157.8 | 529 | 9 | 156.7 | 529 | 10 | 204.2 | 553 | 13 | 1, 2 | 7 |
| 12 | 0.9 | 0.6 | 158.5 | 417 | 7 | 136.4 | 321 | 7 | 166.1 | 291 | 9 | 3, 4 | 7 |
| 12 | 0.9 | 1.0 | 199.3 | 1210 | 11 | 172.9 | 1210 | 1 | 176.1 | 1041 | 1 | 3, 4, 6 | 5 |
| 15 | 0.3 | 0.3 | 123.3 | 516 | 29 | 123.3 | 516 | 29 | 126.4 | 649 | 15 | 3, 4, 5, 6 | 1, 2 |
| 15 | 0.3 | 0.6 | 315.2 | 677 | 21 | 292.9 | 677 | 21 | 379.1 | 453 | 13 | 1, 2 | 7 |
| 15 | 0.3 | 1.0 | 355.1 | 999 | 18 | 355.1 | 999 | 18 | 403.9 | 1735 | 31 | 3, 4, 5, 6 | 1 |
| 15 | 0.6 | 0.3 | 419.7 | 1328 | 35 | 459.6 | 1328 | 35 | 300.4 | 1054 | 19 | 1, 2 | 3, 6 |
| 15 | 0.6 | 0.6 | 223.4 | 803 | 21 | 222.3 | 803 | 25 | 222.3 | 775 | 27 | 1, 2 | 5 |
| 15 | 0.6 | 1.0 | 519.7 | 2987 | 27 | 519.7 | 2987 | 27 | 460.9 | 2465 | 43 | 7 | 1 |
| 15 | 0.9 | 0.3 | 620.1 | 1373 | 19 | 620.1 | 1373 | 19 | 667.3 | 1579 | 5 | 1, 2 | 7 |
| 15 | 0.9 | 0.6 | 1107.6 | 7917 | 9 | 1109.5 | 7917 | 9 | 1187.5 | 11891 | 17 | 5 | 1 |
| 15 | 0.9 | 1.0 | 519.4 | 1675 | 39 | 524.8 | 1675 | 39 | 471.3 | 1528 | 27 | 1, 2 | 3, 4, 6 |
| 20 | 0.3 | 0.3 | 1744.7 | 6045 | 89 | 1648.4 | 5164 | 7 | 1711.6 | 5901 | 3 | 2 | 1 |
| 20 | 0.3 | 0.6 | 2556.8 | 4741 | 439 | 2412.3 | 3532 | 439 | 2658.9 | 13353 | 395 | 2 | 1 |
| 20 | 0.3 | 1.0 | 1018.8 | 1768 | 27 | 1059.8 | 2447 | 27 | 1555.9 | 5642 | 49 | 1 | 3 |
| 20 | 0.6 | 0.3 | 2501.8 | 8817 | 63 | 2017.2 | 3971 | 63 | 1915.5 | 17588 | 133 | 7 | 2 |
| 20 | 0.6 | 0.6 | 1765.1 | 4412 | 57 | 1529.2 | 2701 | 57 | 1694.6 | 10280 | 51 | 6 | 2 |
| 20 | 0.6 | 1.0 | 1134.0 | 1567 | 620 | 1059.2 | 1567 | 620 | 944.0 | 5505 | 525 | 7 | 2 |
| 20 | 0.9 | 0.3 | 3020.0 | 6250 | 107 | 3383.4 | 6250 | 107 | 3177.3 | 24231 | 141 | 2 | 4 |
| 20 | 0.9 | 0.6 | 2353.6 | 4240 | 317 | 2511.4 | 4019 | 317 | 1564.5 | 9150 | 301 | 7 | 2 |
| 20 | 0.9 | 1.0 | 1026.5 | 1879 | 607 | 1938.2 | 2941 | 1554 | 1351.3 | 1697 | 548 | 3 | 6 |

the random nature of our separation algorithms, in some cases implementation 6 is beaten by another one. However, this fact is observed very rarely and does not change the conclusion that implementation 6 is a reliable option to obtain tight LP relaxations for formulation (1).

The last two columns of Table 5 show a comparative analysis of implementations 2–5. Specifically, the results showed that implementations 2 and 5 perform the best and the worst, respectively. Tables 4 and 5 also highlight that implementation 2 always performs better than implementation 1, and that implementations 3 and 4 perform at least as good as implementation 1 in most of the cases. This fact suggests that valid inequalities (2)–(4) individually constitute sharp cuts for formulation (1): they strengthen the formulation more than Xpress proprietary cut strategy does.

Tables 6 and 7 compare the number of nodes explored in the branch-and-cut tree. Although implementations 1 and 2 appear to explore less nodes, the results do not exhibit a clear pattern. Possibly, this is due to our interference with the branch-and-cut implementation of Xpress Optimizer. Apparently, the Xpress proprietary cuts and search strategy help to reduce the size of the branch-and-cut tree, although this does not automatically implies a quick convergence to the optimum.

## 5. Conclusion

In this paper we considered the Partitioning-Hub-Location-Routing Problem (PHLRP), which is a hub location problem with graph partitioning and routing features. The problem mainly arises from the deployment of an Internet routing protocol called Intermediate System–Intermediate System (ISIS), see Retana and White [4] and Özsoy et al. [5], but it also finds applications in the strategic planning of LTL ground freight distribution systems (see Crainic [2], for a recent survey). We introduced a mixed integer programming formulation for its exact solution and we provided families of strengthening valid inequalities. Computational experiments showed the effectiveness of our valid inequalities, which significantly improve the solution time of the Xpress Optimizer solver. Moreover, the valid inequalities yield tight lower bounds when used in a cutting plane algorithm. The formulation can tackle instances of the PHLRP containing up to 20 vertices, which is still far from real-life problem sizes. Nevertheless, our approach offers important prospects for solving large real-life instances of the PHLRP.

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